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Second-order structure function scaling derivation from the Euler and magnetohydrodynamic equations

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An anomalous scaling paradigm that has recently come to be canonical has two features limiting its range of applicability: The driving and driven fields are separated dyamically and the driving field statistics is prescribed, in terms of the (inertial subrange) scaling of its second-order structure functions and of white-noise statistics in time. Then the spectrum of scaling exponents for the driven field, scalar or vector, depends parametrically on the driving. Here, the coupling of turbulent vorticity to the driving velocity field is considered. Using simple approximations and no white-noise statistics assumption, equations are derived for the evolution of two-point second-order correlations. The turbulent magnetohydrodynamic (MHD) case is treated in an analogous fashion. In the neutral case, the kinematic coupling between vorticity and velocity leads to a unique prediction for the scaling exponent of the second-order structure functions of the two turbulent fields. The velocity scaling exponent estimate is $\zeta_2 = 3^{1/2} - 1 \approx 0.732$, i.e., close to experimental data. Unlike Kolmogorov scaling, this result is systematically derived from the Euler equations. The analogous scaling of MHD fields is now treated beyond the dynamo theory approximation. In contrast to the uniqueness found in the neutral case, predicted MHD scalings depend on one parameter, similar to the ''plasma beta'' parameter β_T relating kinetic to magnetic energy. The nature of predicted dependence of inertial-range scaling exponents on β_T agrees with an observed dichotomy between high-β_T and low-β_T turbulence regimes.

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I. INTRODUCTION

Basic notions and assumptions are borrowed from three schools of turbulence modeling, in order to build a quasilinear theory for *second-order* correlations of *solenidal* vector fields which are *coupled* to turbulent velocity in such a way that they influence directly its statistics. The theory is required to (1) predict the inertial-range scaling of second-order structure functions of velocity and vorticity simultaneously; (2) be systematically derived from the dynamical equations, without adjustable degrees of freedom; and (3) account for statistical feedback due both to nonlinearty and to nonlocality induced by the solenoidal projection involved in the momentum equations.

This paper is limited to second-order correlations because these are of greatest interest in applications and also because much more effort and perhaps a different approach would be required to deal systematically with higher-order structure functions.

The background, motivation, and outline of the proposed approach are given in the remainder of this section. The presented result is argued to be a step in overcoming several hurdles that have so far kept one of the major problems, that of anomalous scaling of turbulent velocity, in turbulence theory unsolvable. The paper contains only analytic computations. The necessary background from tensor calculus, as used in homogeneous isotropic turbulence, is presented in Sec. II. It contains all kinematical relations needed later in Sec. IV, in particular, the relation between scaling exponents of vorticity and velocity correlations. An important point in Sec. II is the argument that correlation decay and structure function growth in the inertial range are given by the same exponent. The equation for the steady-state vorticity autocorrelation is derived in Sec. III employing two closure assumptions. These two sections cover, respectively, the kinematic and dynamical aspects of the problem. Their results are combined in Sec. IV, where the ensuing relations between scaling relations are derived. Points of interest beside the scaling exponent of isotropic velocity structure functions (in Sec. IV A), are also the parallel between neutral and MHD dynamics (in Secs. IV B and IV C) and the scaling of helical components of correlations (in Sec. IV C). The results and derivations are interpreted and discussed in Sec. V.

A. Three modeling approaches to developed turbulence

Estimates of turbulence statistics fall into several large categories, of which three are of interest here. The first is a class of closure theories that provide evolution equations for *two-point* correlations. These are based on assumptions of quasi-Gaussianity for low-order statistics of turbulent velocity fields and on a corresponding low-order closure based on some ''perturbative'' approach (Ref. [1] critically reviews the early work on that approach). By construction these closures are limited only to low-order moments and neglect the intermittency due to flow structure. Their subject is the flow of energy (Ref. [2] provides a classical example and set of references). Their standard formulation is for homogeneous turbulence, in terms of averaged Fourier mode interactions.

The second category is that of ''structural'' models treating small-scale vorticity fluctuations as *passively driven* by larger-scale motions. Their natural formulation is in terms of physical-space flow configurations, at least for the large-scale motions which are modeled as simply as possible. The nonlinear *feedback* mechanism remains an issue open for modeling and discussion. One extreme of the ''structural'' approach is represented by the exact solutions of the Burgers vortex [3] kind, in which *dissipative-scale* flow structure is important and feedback is neglected. The statistical aspect has always been at least part of the motivation for work on the "structural" approach, including classical solutions for vortex tubes [3] and vortex sheets [4].

At the opposite extreme, the effect of flow structures is described only in terms of high-order turbulence statistics. In the "anomalous scaling" paradigm, their effect is measured as deviation in power laws, by which structure functions of turbulent fields scale in space, from theoretical estimates of corresponding scaling laws for "structureless" random fields. The variety of models for such deviations pertinent to different turbulent fields constitutes the third "scaling" category.

Successful two-point closure models are compatible with the best known among two-point statistics, the Kolmogorov scaling for the isotropic kinetic energy spectrum, $E(k) \propto k^{-5/3}$, or alternatively, for the second-order velocity structure function, $S^V(2|r) \propto r^{2/3}$, where $r = (\boldsymbol{r}\cdot\boldsymbol{r})^{1/2}$ is the (inertial-range) separation between two points at which the velocity is measured. But they are by construction unable to address anomalous scaling. Successful "structural" models, combining statistical and deterministic elements, are also compatible with the classical $E(k)$, as shown first by Lundgren [5]. Relatively successful "scaling" models tend to involve heuristics about the physical-space structure of turbulent fields, which was the basic ingredient in the "structural" models. The She-Leveque model [6] may be an example. In particular, they predict all $S^V(2|r) \propto r^{\zeta_2}$ with ζ_2 slightly larger than 2/3, a tendency established also by actual measurements.

B. Anomalous scaling of velocity correlations

There is a large number of publications concerning the measurement of scaling exponents in Navier-Stokes turbulence and producing models to fit these observations. Velocity structure functions scale with different powers ζ_q depending on their order q:

$$S_a^V(q|\boldsymbol{r}) = \langle |v_a(\boldsymbol{x}+\boldsymbol{r}) - v_a(\boldsymbol{x})|^q \rangle_{\boldsymbol{x}} \sim r^{\zeta_q^v}. \tag{1}$$

Velocity components $a=1, 2, 3$, are taken with respect to the direction of $\boldsymbol{r}$, which will be assumed to correspond to $a=1$. A *dimensional isotropy* assumption is implicit when the $S_a^V(q|\boldsymbol{r})$ dependence is simplified to a scaling law in r. $S_1^V(q|r)$ is called the (order q) *longitudinal* velocity structure function and $S_2^V(q|r) = S_3^V(q|r)$ the corresponding *transverse* structure function. A *componental isotropy* assumption is implicit in the latter equality. Whether both longitudinal and transverse structure functions of the same order scale with precisely the same exponent, as implicit in the notation of Eq. (1), remains an open issue. Under certain conditions the theoretical prediction is that they do, while measurements are not conclusive. This issue is beyond our present range of interest; some related references are briefly discussed in [7].

The *anomalous scaling* issue is about measuring and rationalizing the deviation of these exponents from the $\zeta_q^v = q/3$ scaling. The famous "4/5 law" giving $\zeta_3 = 1$, complete with explicit proportionality constant in the scaling law, is an exact result due to Kolmogorov. Except for this special case $q=3$, the above Kolmogorov scaling for ζ_q^v is based only on a dimensional argument and is known to fail experimentally. Representative experimental and numerical results can be found, e.g., in Refs. [8–10]. As for models, it was already mentioned that incorporating more turbulence *phenomenology* can improve model predictions. Below are discussed theories based on the explicit use of the dynamical equations, rather than only on heuristics, for inertial-range scaling exponents of passive scalar and "passive magnetic" [magnetohydrodynamic (MHD) dynamo regime] fields. But such fields do not feed back on the driving velocity field whose statistics are (conveniently) prescribed in such models.

To the author's knowledge (see also [11]) there is no velocity scaling theory for ζ_q^v with $q \neq 3$, derived by explicit use of the incompressible Navier-Stokes equations (NSE). The main result of the present study [Eq. (52) below] is an analytic derivation of the isotropic velocity scaling exponent ζ_2^v based only on two explicit, qualitative, approximation assumptions and, on the *explicit* use of the dynamical equations.

C. Rigorous theories of anomalous scaling

Recently, the theory of anomalous scaling of structure functions of passively advected fields based on dynamical equations has enjoyed great progress. Much has been established not only for the classical case of isotropic turbulence, but also for the scaling of anisotropic components of two-point statistics. A recent overview [12] of the passive *scalar* field case contains the set of basic results and references. Further general results can be found in [13]. The scaling of a passive *vector* field is more complicated to compute. The scaling of second-order correlations of a general anisotropic passively advected (MHD dynamo) vector field was reported in [14]. The isotropic case was solved [15] earlier. (The earliest works on dynamo theory, for which [16] serves as an example and reference source, are concerned with the average of the magnetic field itself, i.e., with single-point, first-order statistics. There are, of course, both fundamental and technical similarities with the case of two-point correlations. A quite general viewpoint and list of references is offered by [17].) Despite this progress, there are two aspects in which the literature on anaomalous scaling remains unsatisfactory.

First, the advecting velocity field is assumed to have a white-in-time autocorrelation. This means that the *passively* advected field has much longer memory than the *actively* advecting velocity field. Such an assumption is of rather restricted relevance as a physical model. Except for the case of a scalar diffusivity much larger than viscosity, which is of no interest in our inertial-range discussion, the scalar field correlation scales should be (commensurate) functions of those of the velocity field. The reason why the white-noise model was first introduced by Kraichnan [18] and then persisted in the literature, sometimes slightly modified as, e.g., in [19] and [16], is that it greatly facilitates analytic treatment.

Second, analytic results concern only passively advected fields. The NSE that governs the driving turbulence itself has remained largely without treatment, presumably because of two technical difficulties related to the particular kind of *nonlinearity* present in the underlying dynamical equations. (1) A quadratic *coupling* between the driven and driving fields, precluding any direct approach by the linear theories mentioned above that yield elaborate results for passive fields; (2) the *nonlocality* of solenoidal projection, i.e., the presence of a "pressure term."

Even if boundary and curvature effects are neglected (as usual), nonlocality remains in any of the forms (advection, vorticity, velocity potential) in which NSE may be cast. In a recent response [20] to problem 2, anomalous scaling was studied in a model equation including a pressure term and allowing for anisotropy. But even there, the *advecting* velocity was white noise and its statistics were prescribed; only those of a separate *advected* field were solved for. The difficulty of treating the pressure term was explicitly emphasized in that work, which remained confined to the two-dimensional case in order to alleviate some of the difficulty. The present work was motivated by discussions with the authors of that model. Theory can actually be advanced in both aspects pointed out as unsatisfactory while the two mentioned technical difficulties are avoided rather than attacked.

D. Quasilinear theories

Several recent attempts to model the small scales of turbulence have been very successful in capturing at least the main qualitative features of statistically steady turbulent flows, including two-dimensional (2D) isotropic turbulence [21,22] and 3D wall turbulence [23,24]. The essence of these quasilinear, or rapid-distortion (RDT) approaches is to model small-scale vorticity as passively advected by the velocity field. An obvious motivation is that in flows with a developed inertial range the peaks of energy and enstrophy spectra are located beyond the opposite ends of that range and thereby widely separated. Since energy flows in 3D towards smaller scales, where the vorticity peak is, it can be assumed that the field structure of vorticity is irrelevant to the spatial distribution of energy, which is dominated by the large-scale flow eddies. But vorticity is coupled to velocity, both kinematically and dynamically, so there must be at least a *statistical* mechanism for a feedback. Different publications resort to a variety of conceptual models of this feedback (including the no-feedback case) and to a corresponding variety of formalisms.

In this paper, the spatial *cross-correlations* between velocity and vorticity is assumed to be *negligible* compared to the *product* of velocity and vorticity *autocorrelations*. This is made only for the purpose of approximating two-point, low-order, inertial-range correlations. It is essentially a quasilinear assumption motivated by the above separation-of-scales argument. Solving for the vorticity scaling allows one to avoid any consideration of pressure and pressure-velocity correlations: While the equation for the second-order velocity autocorrelations involves the nonlocality of the NSE through a pressure-strain correlation, that for the *vorticity* autocorrelations involves instead an implicit coupling of vorticity and velocity autocorrelations. The latter is a kinematic relation, leading to a simple relation between the scaling exponents of the two autocorrelations.

The evolution equation for the vorticity correlation tensor is in a form identical with that of the MHD dynamo formalism. A prediction of isotropic magnetic-field correlation scaling in dynamo turbulence due to Ref. [15] gives a *spectrum* of values depending on the scaling of the driving velocity field. It agrees with the isotropic MHD scaling prediction in Sec. IV B below. Unlike dynamo theory, however, the present paper deals with the coupling between MHD current and magnetic-field statistics in parallel with the vorticity-velocity coupling. The magnetic-field feedback through the Lorenz force term is included, and magnetic energy is no longer assumed negligible. This leads to a qualitative agreement with an important feature observed in MHD turbulence — a dichotomy between high-β_T and low-β_T regimes.

A major concern of this article is to relax the unphysical assumption imposed on the "driving velocity" in Kraichnan's model, or its technical substitute in the form of a "linearization assumption," found in various guises in [23,21,20], and elsewhere. The linearization approach in those works models turbulence as a two-phase fluid system, wherein the "small-scale fluid" is passively advected by the "driving fluid." To avoid the need to justify and then carry along a two-phase model, symmetries of the dynamical equation are used here to make important truncations and then only steady states are sought.

Some closure assumptions are unavoidable. The neglect of cross correlations mentioned already is supplemented by a neglect of fourth order cumulants, as in classical closure theories. Both assumptions can be given some empirical support for second-order statistics over the inertial range of length scales, which is precisely our present subject.

II. KINEMATIC RELATIONS

Let $\boldsymbol{v}(\boldsymbol{x},t)$ and $\boldsymbol{\omega}(\boldsymbol{x},t)$ denote, respectively, the velocity and vorticity fields. By definition,

$$\nabla\cdot\boldsymbol{v}=\nabla\cdot\boldsymbol{\omega}=\boldsymbol{0},\qquad \boldsymbol{\omega}=\nabla\times\boldsymbol{v}. \tag{2}$$

The corresponding second-order, two-point, autocorrelation tensors $\boldsymbol{V}$ and $\boldsymbol{\Omega}$ are defined by

$$V_{ab}(\boldsymbol{r},t)=\langle v_a(\boldsymbol{x},t)v_b(\boldsymbol{x}+\boldsymbol{r},t)\rangle_{\boldsymbol{x}} \tag{3}$$

and similarly for $\Omega_{ab}(\boldsymbol{r},t)$. Averaging is over ensemble and also, as indicated, over the flow domain. When the length scales of the domain and the forcing are much larger than the scales of interest, these correlations tend to become isotropic, and will hereafter be assumed such.

In the literature on anomalous scaling, the customary exposition is not in terms of "bare" correlations such as Ω_{ab}, but in terms of structure functions (SF). For example, the second-order velocity SF is

$$\begin{aligned}S^V_{ab}(\boldsymbol{r})&=\langle[v_a(\boldsymbol{x}+\boldsymbol{r})-v_a(\boldsymbol{x})][v_b(\boldsymbol{x}+\boldsymbol{r})-v_b(\boldsymbol{x})]\rangle_{\boldsymbol{x}}\\&=V_{ab}(\boldsymbol{0})-V_{ab}(\boldsymbol{r}),\end{aligned} \tag{4}$$

using homogeneity in the second equality.

To recover the notation of Sec. I, let

$$S_a^V(2|r)=S_{aa}^V(r)$$

with no index summation. In what follows, summation on repeated indices is assumed throughout and the following shorthand notations are used:

$$r=(r_j^2)^{1/2},\qquad \partial_j=\partial/\partial r_j,$$

$$\xi_a=r_a/r=\partial_j r_a.$$

Note $\xi_j^2=1$ and $\partial_b\xi_a=(\delta_{ab}-\xi_a\xi_b)/r$, whence $\partial_j\xi_j=2/r$ and $\partial_b(\xi_a\xi_b)=2\xi_a/r$.

A. Representation of isotropic correlations

Assuming isotropy say for $\boldsymbol{V}$, its classical physical-space representation in 3D is

$$V_{ab}(\boldsymbol{r},t)=\delta_{ab}g_V(r,t)+\xi_a\xi_b[f_V(r,t)-g_V(r,t)],\tag{5}$$

$$g_V(r,t)=r^{-2}\partial_r r^2 f_V(r,t).\tag{6}$$

Due to isotropy, dependence on distance can be factored into f_V and g_V. Analogous relations hold for isotropic $\boldsymbol{\Omega}$, with corresponding f_Ω and g_Ω.

Treating homogeneous statistics rigorously, one necessarily works in a domain periodic in 3D space. The existence of well-behaved Fourier transforms for all considered fields will be assumed. Then $\boldsymbol{V}$ and its Fourier transform $\hat{\boldsymbol{V}}$ can be represented by

$$\hat V_{ab}(\boldsymbol{k},t)=\hat P_{ab}(\boldsymbol{k})\hat Q_{\hat V}(k,t),\tag{7}$$

$$\hat P_{ab}(\boldsymbol{k})=\delta_{ab}-k_ak_bk^{-2},\tag{8}$$

$$g_V(r,t)=-r^{-1}\partial_r r\partial_r Q_V,\tag{9}$$

$$f_V(r,t)=-2r^{-1}\partial_r Q_V,\tag{10}$$

with $\hat Q_{\hat V}(k)$ and its inverse transform $Q_V(r)$ being well-behaved scalar functions. Analogous relations hold for $\hat Q_{\hat\Omega}$. From Eqs. (2) and (7), using the alternating tensor notation ϵ_{abc} in $(\boldsymbol{a}\times\boldsymbol{b})_j=\epsilon_{abj}a_mb_n$, one obtains $\hat P_{ab}(\boldsymbol{k})\hat Q_{\hat\Omega}(k)=k_ik_j\epsilon_{aim}\epsilon_{bjn}\hat P_{mn}(\boldsymbol{k})\hat Q_{\hat V}(k)$, which can be simplified and transformed to physical space:

$$\hat{\boldsymbol{P}}(\boldsymbol{k})\hat Q_{\hat\Omega}(k,t)=\hat{\boldsymbol{P}}(\boldsymbol{k})k^2\hat Q_{\hat V}(k,t),\tag{11}$$

$$\boldsymbol{P}(\boldsymbol{r})Q_\Omega(\cdot,t)=(-\nabla^2)\boldsymbol{P}(\cdot)Q_V(\cdot,t),\tag{12}$$

$$\nabla^2\boldsymbol{PQ}=\boldsymbol{Q}-\boldsymbol{\nabla\nabla}\cdot\boldsymbol{Q},\tag{13}$$

$$c_\Omega r^2+Q_\Omega(r)=-\nabla^2Q_V(r)=-r^{-2}\partial_r r^2\partial_r Q(r).\tag{14}$$

Here $\boldsymbol{I}$ is the 3D unit matrix and c_Ω is some constant. $\boldsymbol{P}$ is kept on the left-hand side in Eq. (12) since it has a nontrivial kernel: Span $(r^2\boldsymbol{I})$. A formal application of Eq. (12), neglecting the boundary conditions needed for proper inversion of $-\nabla^2$, will only define $Q_\Omega(r)$ up to an arbitrary multiple of r^2, i.e., $c_\Omega\neq 0$ in Eq. (14).

B. Scaling range isotropic form

Here we consider only the "scaling range" $0<r_d\ll r\ll r_L$, where r_d is a dissipative length scale and r_L is an "energy containing" scale, comparable to or smaller than the box size. In this case it is possible and advantageous to neglect boundaries in r.

Correlations are expected to decay with distance in physical space. In the scaling range, the decay is assumed algebraic. Such scaling is expected to set in when statistics relax to a time-independent state. Denote the decay exponent of $\boldsymbol{V}$ by $\tilde\lambda>0$. Then

$$g_V=r^{-\tilde\lambda_V},\qquad f_V=r^{-\tilde\lambda_V}2/(2-\tilde\lambda_V),\tag{15}$$

$$Q_V=-r^{2-\tilde\lambda_V}/(2-\tilde\lambda_V)^2,\tag{16}$$

up to a constant multiplier. Boundaries have been neglected here, and both integrations producing Q_V from g_V have been performed between r and ∞.

From Eqs. (16) and (14), requiring decay of $\boldsymbol{\Omega}$, one obtains the following analog of Eq. (16):

$$Q_\Omega/\Lambda_Q(\tilde\lambda_\Omega)=-r^{2-\tilde\lambda_\Omega}/(2-\tilde\lambda_\Omega)^2,\tag{17}$$

$$\Lambda_Q(a)=a^2(a-3)/(a-2),$$

$$\tilde\lambda_\Omega=\tilde\lambda_V+2.\tag{18}$$

With $\boldsymbol{V}\sim r^{-\tilde\lambda_V}$ decaying, Eq. (4) shows that S^V has to grow with r. The standard assumption is that there is a scaling range where $S^V\sim r^{\lambda_V}$ with $\lambda_V>0$. The analogs of Eqs. (5) and (15) are then

$$S_{ab}^V(\boldsymbol{r})=S_0^V r^{\lambda_V},P_{ab}(\lambda_V;\boldsymbol{\xi}),\tag{19}$$

$$P_{ab}(\lambda;\boldsymbol{\xi})=\delta_{ab}-\xi_a\xi_b\lambda/(2+\lambda),$$

$$f_{SV}(r)=r^{\lambda_V}2/(2+\lambda_V),\tag{20}$$

$$g_{SV}(r)=r^{\lambda_V},$$

and similarly for $\boldsymbol{S}^\Omega$. It is noted that $\partial_j^2P_{ab}(\lambda+2;\boldsymbol{\xi})$ cannot be written as $\Lambda_P(\lambda)P_{ab}(\lambda;\boldsymbol{\xi})$.

Later it will be seen that the equilibrium state equation is formally the same for both $\boldsymbol{\Omega}$ and $\boldsymbol{S}^\Omega$. Here it will only be verified for any couple $\boldsymbol{V}$ and $\boldsymbol{S}^V$ related by Eq. (4), that

$$\lambda_V=-\tilde\lambda_V\tag{21}$$

and, of course, $\lambda_\Omega=-\tilde\lambda_\Omega=2+\lambda_V$, in an appropriate scaling range. The argument is the same for g_V,g_{SV} couples and f_V,f_{SV} couples. Suppressing the suffix $\cdot_V$, consider $g\sim r^{-\tilde\lambda}$ and $g_S\sim r^\lambda$. From Eq. (4) $g_S(r)=g_S(0)-g(r)$. Up to an irrelevant prefactor, this reads in the scaling range as $g_S(r)=c-r^{-\tilde\lambda}$, where $0<c=\mathcal{O}(1)$ is a constant. Now r

$\ll 1$ in the scaling range, rendering the rational approximation $g_S(r)=r^{\tilde{\lambda}}/(1+r^{\tilde{\lambda}}c)$ acceptable. Comparison with Eq. (20) gives Eq. (21).

The above argument suggests to modify Eq. (20) into

$$g_{SV}(r)=g_{SV}(0)-r^{-\lambda}, \tag{22}$$

$$f_{SV}(r)=g_{SV}(0)-r^{-\lambda}2/(2-\lambda), \tag{23}$$

$$Q_{SV}(r)=Q_{SV}(0)-g_{SV}(0)(r/2)^2+r^{2-\lambda}/(2-\lambda)^2, \tag{24}$$

and similarly for $g_{S\Omega}, f_{S\Omega}$, and $Q_{S\Omega}$. In particular,

$$Q_{S\Omega}=\lambda^2\{g_{SV}(0)+g_{S\Omega}(0)(r/2)^2-r^{-\lambda}[1/(2-\lambda)+1]\}.$$

In the special case $\lambda=2$, the velocity correlations are not anomalous and the $\xi_a\xi_b$ term is absent from $\boldsymbol{V}$ and $\boldsymbol{S}_V$. If $\lambda=0$, the same assertions apply to the vorticity field instead. The terms representing anomalous scaling are absent ($f=g$) in those cases when their denominators would vanish.

C. Representation of helicity

Relaxing the isotropy constraint on a tensor means that its scale dependence can no longer be expressed only using the trace $\hat{Q}(k)$ or $Q(r)$. Since $\mathrm{Tr}[\hat{Q}(\boldsymbol{k})]$ is invariant to rotations of the $\boldsymbol{k}$-coordinate system, a minimum extension is to include another such invariant. The simplest among few choices is

$$\hat{h}_{ab}(\boldsymbol{k},t)=\epsilon_{abj}\hat{\imath}k_j\hat{h}(k,t), \tag{25}$$

$$h_{ab}(\boldsymbol{r},t)=H_{ab}(\boldsymbol{\xi})\tilde{h}'(r,t), \tag{26}$$

$$H_{pq}(\boldsymbol{\xi})=\epsilon_{pqj}\xi_j. \tag{27}$$

Here $\hat{h}$ is the Fourier transform of $\tilde{h}$, $\tilde{h}'=\partial\tilde{h}/\partial r$, $\hat{\imath}$ is the imaginary unit and ϵ_{mnj} is the alternating tensor.

Unlike $\mathrm{Tr}(\hat{Q})$, $h_{ab}(\boldsymbol{r})$ is not reflexion invariant. Strictly speaking, helicity is related only to the real part of $\hat{h}(k)$, but both $\hat{h}(K)$ and $h=\tilde{h}'(r)$ will be loosely referred to as "helicity." Incompressibility does not restrict h: $\epsilon_{abj}\partial_a\xi_b h=0$ always holds. The relation between scaling of helical and isotropic parts of correlations will be considered later on. It may be helpful while following some computations in Sec. IV to note that

$$\epsilon_{ajm}\epsilon_{jbn}=\delta_{ab}\delta_{nm}-\delta_{an}\delta_{bm}, \tag{28}$$

$$H_{ab}(\boldsymbol{\xi})\xi_a\xi_b=0,$$

$$\partial_j^2 r^{-\sigma}H_{ab}(\boldsymbol{\xi})=\Lambda_H(\sigma)r^{-\sigma-2}H_{ab}(\boldsymbol{\xi}), \tag{29}$$

$$\Lambda_H(a)=\sigma(\sigma-1),$$

$$\partial_m\partial_n r^{\sigma}[H_{am}(\boldsymbol{\xi})H_{bn}(\boldsymbol{\xi})]=0, \tag{30}$$

$$\partial_m\partial_n r^{\sigma}[H_{ab}(\boldsymbol{\xi})H_{mn}(\boldsymbol{\xi})]=0, \tag{31}$$

$$\partial_m\partial_n r^{\sigma}[P_{am}(\lambda;\boldsymbol{\xi})H_{bn}(\boldsymbol{\xi})+P_{bn}(\lambda;\boldsymbol{\xi})H_{am}(\boldsymbol{\xi})]=(\sigma-1)r^{\sigma-2}(H_{ab}+H_{ba})=0, \tag{32}$$

$$\partial_m\partial_n r^{\sigma}[P_{ab}(\lambda;\boldsymbol{\xi})H_{mn}(\boldsymbol{\xi})+P_{mn}(\lambda;\boldsymbol{\xi})H_{ab}(\boldsymbol{\xi})]=[3(\sigma-1)-2(\sigma+2)\lambda]r^{\sigma-2}H_{ab}(\boldsymbol{\xi}). \tag{33}$$

Using Eq. (28), the analogs of Eqs. (11) and (12) are found to be $\hat{h}_{\hat{\Omega}}(k)=k^2\hat{h}_{\hat{V}}(k)$ and $h_\Omega(r)=-\partial_j^2 h_V(k)$. In the scaling range, the relation between h_V and h_Ω is then given by Eq. (29): If $h_V=r^{-\tilde{\lambda}_V}$ then $h_\Omega=\Lambda_H(\tilde{\lambda}_\Omega)r^{-\tilde{\lambda}_\Omega}$ with $\tilde{\lambda}_\Omega=\tilde{\lambda}_V+2$ as before. However, Λ_H differs from Λ_Q given in Eq. (17).

III. DERIVATION OF A STEADY-STATE EQUATION

Our subject is inertial range scaling, so the dissipative term in the vorticity evolution equation is omitted:

$$\partial_t\boldsymbol{\omega}=(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega}. \tag{34}$$

This corollary of the Euler equations is formally identical with the inviscid MHD equation for the magnetic induction $\mathbf{b}$, but the latter must be coupled with either the momentum or vorticity equation of MHD. Upon normalization,

$$\partial_t\mathbf{b}=(\mathbf{b}\cdot\boldsymbol{\nabla})\boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{\nabla})\mathbf{b}, \tag{35}$$

$$\partial_t\boldsymbol{\omega}=[(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v}-(\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega}]-[(\mathbf{j}\cdot\boldsymbol{\nabla})\mathbf{b}-(\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{j}], \tag{36}$$

$$\mathbf{j}=\boldsymbol{\nabla}\times\mathbf{b},\ \boldsymbol{\nabla}\cdot\mathbf{b}=\boldsymbol{\nabla}\cdot\mathbf{j}=\mathbf{0}. \tag{37}$$

The analogy between Eqs. (35),(37) and (2),(34) is exploited in dynamo theory by neglecting the Lorentz term in Eq. (36), so that it becomes the same as Eq. (34), on the assumption that $|\mathbf{j}|\ll|\boldsymbol{v}|$.

A. Equations for steady-state correlations

From Eq. (34),

$$\boldsymbol{\omega}(x,t)-\boldsymbol{\omega}(x,0)=\int_0^t(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v}(x,\tau)d\tau-\int_0^t(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega}(x,\tau)d\tau.$$

Repeating this for $\boldsymbol{\omega}(x+r,t)$ and recalling Eq. (3), one obtains the spatially local equation

$$\Omega_{ab}(r,t)=\Omega_{ab}(r,0)+\int_0^t\Omega_{ab}^{[1]}(r,\tau)d\tau+\int_0^t\int_0^t\Omega_{ab}^{[2]}(r,\tau_1,\tau_2)d\tau_1 d\tau_2, \tag{38}$$

$$\Omega_{ab}^{[1]}(r,t_1)=\langle\omega_a(o,0)\omega_j(+,1)v_{b,j}(+,1)\rangle-\langle\omega_a(o,0)v_j(+,1)\omega_{b,j}(+,1)\rangle+\langle\omega_b(+,0)\omega_j(o,1)v_{a,j}(o,1)\rangle-\langle\omega_b(+,0)v_j(o,1)\omega_{b,j}(o,1)\rangle, \tag{39}$$

$$\Omega_{ab}^{[2]}(r,t_1,t_2)=\langle\omega_j(o,1)v_{a,j}(o,1)\omega_m(+,2)v_{b,m}(+,2)\rangle -\langle\omega_j(o,1)v_{a,j}(o,1)v_m(+,2)\omega_{b,m}(+,2)\rangle +\langle v_j(o,1)\omega_{a,j}(o,1)v_m(+,2)\omega_{b,m}(+,2)\rangle -\langle v_j(o,1)\omega_{a,j}(o,1)\omega_m(+,2)v_{b,m}(+,2)\rangle. \quad (40)$$

Notations like

$$\omega_a(o,\cdot)=\omega_a(x,\cdot),$$

$$\omega_a(+,\cdot)=\omega_a(x+r,\cdot),$$

$$\omega_a(-,\cdot)=\omega_a(x-r,\cdot),$$

used in averages with respect to x, and

$$\omega_a(\cdot,1)=\omega_a(\cdot,t_1),$$

$$\omega_a(\cdot,0)=\omega_a(\cdot,0),$$

$$\omega_{a,j}=\partial_j\omega_a,$$

are introduced for brevity. Using homogeneity and Eq. (2), the term in the first brackets in Eq. (39) can be rewritten as $\partial_j W_{a,bj}(-r)$, where

$$W_{a,bj}(\pm r)=\{\omega_a(\pm,0)[\omega_j(o,1)v_b(o,1)-\omega_b(o,1)v_j(o,1)]\}$$

is antisymmetric in b,j. Starting from the general form

$$W_{abj}=W_{31}\xi_a\xi_b\xi_j+(W_{21}\xi_a\delta_{bj}+W_{22}\xi_b\delta_{ja}+W_{23}\xi_j\delta_{ab}) +(W_{01}\epsilon_{bjp}\xi_a+W_{02}\epsilon_{jap}\xi_b+W_{03}\epsilon_{abp}\xi_b)\xi_p,$$

where all W_{pq} are functions of r and t, application of the incompressibility condition gives $(1+r\partial_j)W_{0j}=0$ whence $W_{0j}=0$ for $j=1,2,3$. Antisymmetry reduces the coefficients of isotropic terms: $W_{a,bj}(-r)=W(r)(\xi_j\delta_{ab}-\xi_b\delta_{aj})$. Thus $\partial_j W_{a,bj}=\delta_{ab}(W'+W/r)-\xi_a\xi_b(W'-W/r)$. The term in the second brackets in Eq. (39) can be rewritten as $\partial_j W_{a,bj}(+r)$. But $W(r)$ is even, so the two terms sum to zero and $\Omega_{ab}^{[1]}=0$ from symmetry considerations only. Since $\Omega_{ab}^{[1]}(r,t_1)=t^2[\Omega_{ab}^{[2]}(r,0,0)+\mathcal{O}(t)]$, Eq. (38) transforms, using homogeneity again, into

$$\partial_t^2\Omega_{ab}(r)=\partial_j\partial_m M_{abjm}(r,t), \quad (41)$$

$$M_{abjm}(r,t)=[\langle\omega_j(o)v_a(o)\omega_m(+)v_b(+)\rangle -\langle\omega_j(o)v_a(o)v_m(+)\omega_b(+)\rangle +\langle v_j(o)\omega_a(o)v_m(+)\omega_b(+)\rangle -\langle v_j(o)\omega_a(o)\omega_m(+)v_b(+)\rangle]. \quad (42)$$

Only single-time correlations appear in Eq. (42).

At this point, an approximation can be made by neglecting fourth-order cumulants or modeling them in terms of the second-order statistics directly solved for in the model. It will reduce all terms in $\boldsymbol{M}(r,t)$ to products of second-order moments. This reduction is in the spirit customary in two-point closure theories dealing only with turbulence spectra, e.g., Kraichnan's direct-interaction approximation or Orszag's eddy-damped quasinormal Markovian closure [1]. The expectation, which can only be justified by comparing theoretical predictions to experimental data, is that flow events leading to a deviation from Gaussian statistics are more or less the signatures of dissipative-scale vorticity structures and therefore not expected to influence significantly the lowest-order inertial-range statistics. Approximations of this quasinormal nature are clearly not applicable when the scaling of high-order structure functions is investigated, because anomalous scaling is a manifestation of the increasing degree of non-Gaussianity of the velocity statistics going downward in the inertial range. Even if limited to second-order statistics only, one is confronted with serious difficulties in modeling adequately the neglected terms in the case of anisotropic turbulence. But our present topic is confined only to second-order isotropic statistics, so making a high-order closure assumption is at least not obviously unreasonable. Hereafter, fourth-order cumulants will be neglected outright. This is the simplest and crudest approximation, notorious for being shown (*a posteriori*) to fail in the spectral theory of isotropic turbulence. How good or bad it is for the present purpose, however, or as a matter of course, for strongly anisotropic turbulence spectral computations, can be found out again *a posteriori*. Before that, there is no compelling reason to introduce any modeling of the neglected cumulants. Several decades of experience in spectral theory have shown that such modeling is rather demanding both technically and in terms of justifying its underlying assumptions and parametric choices.

Although cross correlations entering the result will be only single-point ones, their contribution does not vanish unless $\langle\omega_a(o)v_b(o)\rangle=\langle v_b(o)\omega_b(o)\rangle$. Instead of insisting on such an equality (which obviously does not hold in general), an order-of-magnitude argument can be made, based on experimental observations; it then leads to a second approximation, invoked in order to dispose of the cross correlations for which a separate equation must otherwise be kept along. One such observation by this author concerns statistics from direct numerical simulations (DNS) of 3D isotropic turbulence at 512^3 resolution, for which a detailed account still remains to be published. Fourier spectra of cross correlations were computed and seen to be strongly fluctuating. But their "envelope spectra" can be defined reasonably well over the inertial and dissipative ranges and appear close to, but below the corresponding velocity spectra, by about an order of magnitude. Of course, vorticity spectra are orders of magnitude larger in those ranges and growing with wave number. This suggests to neglect cross correlations as being "small" compared to autocorrelations.

To summarize, the present derivation is based on neglecting all (i) fourth-order cumulants and (ii) cross correlations. Within the applicability range of both (qualitative) assumptions,

$$M_{abmn}[V,\Omega](r)=[\Omega_{ab}(r)V_{mn}(r)+V_{ab}(r)\Omega_{mn}(r)] -[\Omega_{am}(r)V_{nb}(r)+V_{am}(r)\Omega_{nb}(r)]. \quad (43)$$

It is possible to put $\boldsymbol{M}[S^V,S^\Omega]$ instead of $\boldsymbol{M}[V,\Omega]$ into Eq. (41), since $\partial_j\partial_m\boldsymbol{M}(r)=\partial_j\partial_m[\boldsymbol{M}(r)-\boldsymbol{M}(0)]$. It is also possible to put $\boldsymbol{M}[S^V,\Omega]$ instead, substracting from $M_{abmn}(r)$ the terms $\langle\omega_j(o)v_a(o)\omega_m(+)v_b(0)\rangle$ and $\langle v_j(o)\omega_a(o)v_m(+)\omega_b(0)\rangle$ which vanish under the action of ∂_m and then adding the terms $\langle\omega_j(+)v_a(o)v_m(+)\omega_b(+)\rangle$ and $\langle v_j(+)\omega_a(o)\omega_m(+)v_b(+)\rangle$ that vanish under ∂_j.

This nonuniqueness in the choice of statistics invites the use of $\langle V,\Omega\rangle$ as a place holder for any possible couple of "velocity," "vorticity" statistics, in that order, as indicated. Then Eq. (41) suggests the steady-state equation

$$\nabla\nabla:\boldsymbol{M}[V,\Omega]=0. \tag{44}$$

Precisely the same equation would follow from the MHD induction equation if $\boldsymbol{\Omega}$ is to be understood as the magnetic-field autocorrelation, and if the cross correlation between the "driving" velocity and the "driven" magnetic field is neglected on the basis of an analogous justification. The MHD vorticity equation can be treated similarly. The complete set of steady-state equations in the MHD case then reads

$$\nabla\nabla:\boldsymbol{M}[V,B]=0,$$
$$\nabla\nabla:\boldsymbol{M}[V,\Omega]=\nabla\nabla:\boldsymbol{M}[B,J], \tag{45}$$

again subject to possible replacement of $\boldsymbol{V}$ by $\boldsymbol{S}^V$, $\boldsymbol{B}$ by $\boldsymbol{S}^B$, etc., in any occurrence of $\boldsymbol{M}$.

IV. SCALING PREDICTIONS

At the outset, let the scaling of $\boldsymbol{\Omega}\sim r^{-\mu}$ be formally independent from that of $\boldsymbol{V}\sim r^{-\lambda}$, to allow derivations to be reused in the MHD case. For convenience, let

$$\lambda+\mu=\sigma_1,\quad \sigma_1+2=-\sigma_0,$$
$$\lambda/(2+\lambda)+\mu/(2+\mu)=2\sigma_2.$$

With these notations, recalling Sec. II B and ignoring constant factors, the isotropic scaling version of Eq. (44) is

$$\begin{aligned}\partial_j\partial_m M_{abmn}[V,\boldsymbol{\Omega}]/2=&\,\partial_j\partial_m r^{-\sigma_1}[(\delta_{aj}\delta_{bm}-\delta_{ab}\delta_{jm})\\&+\sigma_2(\xi_a\xi_b\delta_{jm}+\xi_j\xi_m\delta_{ab}-\xi_a\xi_j\delta_{bm}\\&-\xi_b\xi_m\delta_{aj})]=[\sigma_1(\sigma_2-1)+\sigma_2]\\&\times[-\sigma_1P_{ab}(\sigma_0;\boldsymbol{\xi})r^{\sigma_0}].\end{aligned} \tag{46}$$

When $\sigma_1=0$ the term $-\sigma_1P_{ab}(\sigma_0)$ is replaced by $2\xi_a\xi_b$. The isotropic eigenvalue relation becomes

$$\lambda/(2+\lambda)+\mu/(2+\mu)=2(\lambda+\mu)/(\lambda+\mu+1). \tag{47}$$

When helicity is taken into account, the scaling forms of the tensor arguments of $\boldsymbol{M}[\boldsymbol{V},\boldsymbol{\Omega}]$ are

$$\boldsymbol{V}(\boldsymbol{r})=Q_Vr^{-\lambda}\boldsymbol{P}(-\lambda;\boldsymbol{\xi})+h_Vr^{-\hat{\lambda}}\boldsymbol{H}(\boldsymbol{\xi}),$$

where Q_V and h_V are now constant, and similarly for $\boldsymbol{\Omega}(\boldsymbol{r})$. Inserting these on the right-hand side in Eq. (45) gives several contributions as follows. From isotropic parts only, Eq. (46) multiplied by Q_VQ_Ω, from helical parts only, terms that vanish due to Eq. (31); from cross multiplication of helical and isotropic parts,

$$\begin{aligned}&\partial_j\partial_m\{Q_Vh_\Omega r^{-(\lambda+\hat{\mu})}\boldsymbol{M}[\boldsymbol{P}(-\lambda;\boldsymbol{\xi})\boldsymbol{H}(\boldsymbol{\xi})]\\&\quad+Q_\Omega h_Vr^{-(\mu+\hat{\lambda})}\boldsymbol{M}[\boldsymbol{P}(-\mu;\boldsymbol{\xi})\boldsymbol{H}(\boldsymbol{\xi})]\}\\&=-[Q_Vh_\Omega r^{-(\lambda+\hat{\mu}+2)}\chi(\lambda,\hat{\mu})\\&\quad+Q_\Omega h_Vr^{-(\mu+\hat{\lambda}+2)}\chi(\mu,\hat{\lambda})]\boldsymbol{H}(\boldsymbol{\xi}).\end{aligned} \tag{48}$$

The form of χ in Eq. (48) results from Eqs. (32) and (33), which imply together with Eq. (44)

$$\partial_j\partial_m\boldsymbol{M}[r^\lambda\boldsymbol{P}(\lambda;\boldsymbol{\xi}),r^\mu\boldsymbol{H}(\boldsymbol{\xi})]=-\chi(-\lambda,-\mu)\boldsymbol{H}(\boldsymbol{\xi}), \tag{49}$$

$$\chi(\lambda,\mu)=(3+2\lambda)(\lambda+\mu)+(3-4\lambda). \tag{50}$$

From Eqs. (44), (46), and (48) follows a "dispersion relation" for $\hat{\lambda}$ and $\hat{\mu}$ in terms of λ and μ:

$$(h_\Omega/h_V)r^{\lambda-\hat{\lambda}}\chi(\lambda,\hat{\mu})+(Q_\Omega/Q_V)r^{\mu-\hat{\mu}}\chi(\mu,\hat{\lambda})=0. \tag{51}$$

Depending on the relation between $\boldsymbol{V}$ and $\boldsymbol{\Omega}$, this equation may either be split in two: $\chi(\lambda,\hat{\mu})=0$ and $\chi(\mu,\hat{\lambda})=0$, or otherwise, it is sure that there is a relation between λ and μ independent from Eq. (51).

A. Scaling of isotropic correlations

Now $\boldsymbol{\Omega}$ may be interpreted as the *vorticity* correlation. Anticipating the outcome, consider $\boldsymbol{M}[S^V,\Omega]$ replacing $\boldsymbol{M}[V,\Omega]$ in Eq. (46), thus making the association $\lambda=\lambda_V$ and $\mu=\tilde{\lambda}_\Omega$. In addition, the constraint $\tilde{\lambda}_\Omega=-(\lambda_V+2)$ holds in view of Eqs. (18) and (21). In that case one of the solutions to Eq. (47) in terms of λ_V is negative, which is unacceptable for a structure function scaling. Thus we are left only with

$$\zeta_2\approx\lambda_V=3^{1/2}-1\approx0.732. \tag{52}$$

Instead of $\boldsymbol{M}[S^V,\Omega]$ it was possible to use $\boldsymbol{M}[V,\Omega]$ or $\boldsymbol{M}[S^V,S^\Omega]$ or $\boldsymbol{M}[V,S^\Omega]$ in Eq. (46), the difference being only that $\tilde{\mu}\to\mu$ or $\lambda\to\tilde{\lambda}$. With these substitutions Eq. (47) remains formally the same, but the additional constraint changes. Only one more physically acceptable solution can be obtained after checking all possibilities: $\lambda_V=(2/3)^{1/2}\approx0.8165$ from $\boldsymbol{M}[V,\Omega]$. In comparison with Eq. (52), it predicts faster decay of correlations with increasing distance r, leaving Eq. (52) as the prediction for actually observable scaling.

Kolmogorov scaling means $\zeta_2=\lambda_K=2/3$. A list of recent experimental results on ζ_2 might include the following estimates: 0.688 ± 0.002 form a 512^3 DNS with $R_\lambda=218$ [8], 0.708 for atmospheric surface layer at $R_\lambda=10\,340$ and error bar illegible [10] but presumably of the same order as above, $\lambda_l=0.70$ and $\lambda_t=0.68$ with differentiation between longitudinal and transverse exponents, for surface layer data [9] with $R_\lambda = 103\,40$–$148\,60$ and error bars "difficult to quantify" but conservatively bounded by ± 0.02. The cited measurements suggest an experimental $\lambda_E = 0.70 \pm 0.01$ if λ_I is used as "isotropic value." The deviation of λ_K and λ_V from this λ_E is -4.7% and 4.6%, respectively.

Thus, prediction (52) is quantitatively as good, or as bad, as the Kolmogorov scaling, but qualitatively better in the sense that it predicts the direction in which the anomalous exponent deviates from λ_K. The overestimate is rather small in view of the approximations made. Indeed, an *ad hoc* modification $\mu = \lambda + 2.062$ would yield $\lambda_V = 0.7002$; this is to be compared with the *ad hoc* substitution of $\langle \Delta v(r)^3 \rangle$ by $\langle |\Delta v(r)|^3 \rangle$ which is empirically $\approx \langle \Delta v(r)^3 \rangle^{1.05}$ according to [10]. Since λ_E are frequently estimated using the extended self-similarity hypothesis that introduces $\langle \Delta v(r)^3 \rangle$, the error of such measurements can thus be comparable, in principle, to its difference from theory.

B. Isotropic MHD

In Eq. (45) a triple of arguments $\{[V,B],[V,\Omega],[B,J]\}$ appears. All triples obtained from it by substituting any number of tensor arguments by their structure function counterparts should be considered, e.g., $\{[S^V,B],[V,\Omega],[S^B,S^J]\}$, $\{[S^V,S^B],[S^V,\Omega],[S^B,J]\}$, etc. Recalling again Sec. II B, each of the three terms in each case can be obtained from Eq. (46) by a triple of appropriate redefinitions of σ_1, σ_2, and σ. The resulting eigenvalue problem is then cast in the form of algebraic equations, given by

$$\rho(\lambda_1^V,\lambda_1^B) = 0, \tag{53}$$

$$\rho(\lambda_2^V,\lambda_3^V) = \beta\rho(\lambda_2^B,\lambda_3^B), \tag{54}$$

together with the definition

$$\rho(a,b) = (a+b+1)[a/(a+2) + b/(b+2)] - 2(a+b), \tag{55}$$

and a set of linear relations between λ_j^V and λ_V, and between λ_j^B and λ_B. (The value of β^{-1} in the present context is related to the MHD "beta" measuring the ratio of total kinetic to total magnetic energy, but the relation is difficult to quantify and need not be universal. To be precise, the second equations in Eqs. (45) and (53) predict $[\beta\Lambda_Q(\lambda_B)/\Lambda_Q(\lambda_V)]^{1/2}$ as the ratio between magnetic and kinetic correlations.) Two possible approaches to such a system are either to solve for all variables, including β, raising the problem of its interpretation, or to require a solution valid for any β, thus splitting the second equation into a pair of independent equations: $\rho(\lambda_2^V,\lambda_3^V) = 0$ and $\rho(\lambda_2^B,\lambda_3^B) = 0$. For any of these approaches, and for each instance of an argument triple, a set of several solutions is obtained. Each solution has to be checked and rejected if it includes imaginary values, or if the sign of either λ_j^V or λ_j^B is not as expected from the imposed relation with λ_V, λ_B, that is, if a structure function scaling turns out negative or a correlation decay exponent turns out positive. In some cases, not a single solution but a one-parameter family of solutions is obtained, which have to be scanned for admissibility at least over the physically reasonable range of either λ_V or λ_B between 0 and say 3. The solution can rarely be represented conveniently in terms of either λ_V or λ_B as free parameter.

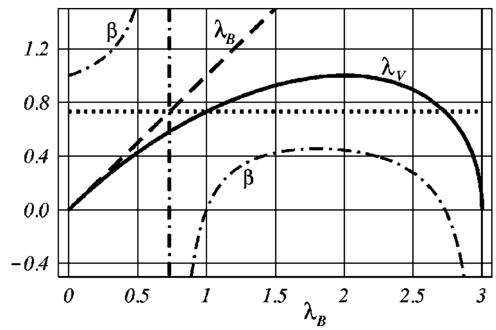

FIG. 1. Isotropic β-dependent MHD scaling; dots, nonmagnetic $\lambda_V = 3^{1/2} - 1$; solid curve, $\lambda_V(\lambda_B)$; dash-dotted, $\beta(\lambda_B)$ with vertical line indicating $1/\beta = 0$.

Clearly the search for acceptable solutions is tedious and thus error prone. To the extent to which this author has been able to deal with it, there are only three kinds of solutions: First, $\lambda_B = \lambda_V = 3^{1/2} - 1$ as in Eq. (52). This solution arises from $\{[S^V,B],[S^V,\Omega],[S^B,J]\}$ and is valid for any β. Second, $\lambda_B = \lambda_V = (2/3)^{1/2}$ as in the second value obtained in the nonmagnetic case. This comes from $\{[V,B],[V,\Omega],[B,J]\}$ and is also valid for any β. Finally, a parameter-dependent family of solutions arises from $\{[S^V,B],[S^V,\Omega],[S^B,J]\}$ when β is allowed to be part of the solution:

$$\lambda_V(\lambda_B) = (\lambda_B + \Delta - 3)/2, \tag{56}$$

$$\beta(\lambda_B) = (\lambda_B/\lambda_V)(1 + \Delta - \lambda_B)(1 + \Delta + \lambda_B)^{-1}, \tag{57}$$

$$\Delta(\lambda_B) = 3(1 + \lambda_B 2/3 - \lambda_B^2/3)^{1/2}, \tag{58}$$

shown in Fig. 1. The solution agrees with the scaling (52) in the limiting case of vanishing velocity field $1/\beta = 0$) as seen at the crossing of dotted and dash-dotted lines in Fig. 1, and of course in the limit of nonmagnetic turbulence ($\beta = 0$). As in the nonmagnetic case, the fixed value $(2/3)^{1/2}$ can be discarded in favor of $3^{1/2} - 1$, which in turn is to be compared to the value of λ_V from the β-dependent solution. The qualitative outcome is a plausible prediction that the same scaling as in nonmagnetic turbulence holds, except for sufficiently "low-beta" flows. Complex behavior is predicted in the β-range where $\lambda_V < \zeta_2 < \lambda_B$.

Due to the equations' symmetry, there is another solution in which λ_V and λ_B exchange roles. This is the solution given by [15]. In it, the magnetic correlation has longer range than the velocity correlation, and β^{-1} is proportional to the MHD "beta" parameter β_T. But the qualitative outcome remains the same. In [15], only Eq. (53) was considered, without coupling to Eq. (54), and no β_T dependence was found. The appearance of β is clearly due to the Lorenz-force coupling.

Other qualitative features of the solution are worth notice.

(i) $\lambda_V < \lambda_B$ throughout.

(ii) For $\lambda_V \leq \zeta_2$, which is the range of interest here, λ_V and λ_B are close to each other. This means $\Lambda_Q(\lambda_B)/\Lambda_Q(\lambda_V)$

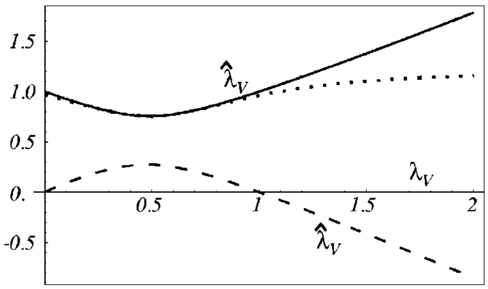

FIG. 2. Solution branches for "helicity" scaling in a vorticity-velocity (or current-magnetic field) dependence (solid line) and a Padé (3:3) approximation to it (dotted line). The unphysical lower branch is shown by a dashed line.

is close to 1 and the use of β from Eq. (57) is sufficient for the present qualitative discussion.

(iii) The extremes $\beta=0$ and $\beta\to\infty$ are present in the solution and exponents λ become arbitrarily close to zero as $\beta\to 1$ (presumably corresponding to "low-β_T" turbulence).

(iv) The range $\beta<0$ is unphysical, while a small range of positive values is not covered by β in this solution.

A more in-depth evaluation of the present result requires comparison with scaling data from MHD turbulence at various ratios of kinetic to magnetic energy. Such comparison remains beyond the scope of this paper.

C. Scaling of "helicity"

When the anisotropic tensors introduced earlier are allowed in Eq. (44), the dispersion relation (51) may be specialized taking either $\boldsymbol{M}[\boldsymbol{V},\boldsymbol{V}]$ or $\boldsymbol{M}[\boldsymbol{S}^V,\boldsymbol{S}^V]$, but not mixed argument couples. This assures $\hat{\lambda}-\lambda=\hat{\mu}-\mu$ and that the ratios in brackets depend only on $\hat{\lambda}$ and λ, through Λ_Q and Λ_H introduced in Eqs. (17) and (29). The result is effectively an equation for $\hat{\lambda}$

$$\Lambda_H(\hat{\lambda}_V)\,\chi(\lambda,\hat{\mu})=\Lambda_Q(\lambda_V)\,\chi(\mu,\hat{\lambda}) \tag{59}$$

with a single parameter λ. Only two solution branches for $\boldsymbol{M}[\boldsymbol{V},\boldsymbol{V}]$ are acceptable at $\lambda=\zeta_2$ in Eq. (52), but one of the branches is unphysical, since for all λ values it gives too low and somewhere even negative $\hat{\lambda}$'s (as seen in Fig. 2) for the scaling of the "helical component" of the velocity structure function. Selecting the upper branch and using λ_V from Eq. (52) gives

$$\hat{\lambda}_V\approx 0.830. \tag{60}$$

As an illustration of the algebra involved, here is the exact solution for $\hat{\lambda}_V$ in that branch:

$$\hat{\lambda}_V(\mu)=[16\mu^4+80\mu^3+2\mu^2(60-\kappa_0)+\mu(158-5\kappa_0)+192+(\kappa_0-1)^2](\kappa_3\kappa_0)^{-1}, \tag{61}$$

$$\kappa_0(\mu)=[\kappa_1+\kappa_3(-3\kappa_2)^{1/2}]^{1/3},$$

$$\kappa_3(\mu)=3(7+2\mu),$$

$$\kappa_1(\mu)=8\mu^3(8\mu^3+60\mu^2+138\mu+119)-5(294\mu^2+1569\mu+115), \tag{62}$$

$$\kappa_2(\mu)=8\mu^3(32\mu^8+208\mu^7+944\mu^6+2540\mu^5+2948\mu^4+3265)-29195\mu^2+3564\mu+5184. \tag{63}$$

To enable some of the computer algebra in the case discussed next, this solution had to be Padé approximated, as shown by a dotted line in Fig. 2. Calculations and plotting were done using MATHEMATICA.

When anisotropic tensors are allowed in the MHD case, the relation between λ_V, λ_B, and β through Eqs. (56)–(58) must be accounted for. From Eqs. (45) and (51) follow

$$(h_B/h_V)r^{\hat{\lambda}_V-\lambda_V}\chi(\lambda_V,\hat{\lambda}_B)+(Q_B/Q_V)r^{\hat{\lambda}_B-\lambda_B}\chi(\mu_B,\hat{\lambda}_V)=0, \tag{64}$$

$$\begin{aligned}[]&[(h_\Omega/h_V)r^{\lambda_V-\hat{\lambda}_V}\chi(\lambda_V,\hat{\lambda}_\Omega)\\ &\quad+(Q_\Omega/Q_V)\,r^{\lambda_\Omega-\hat{\lambda}_\Omega}\,\chi(\lambda_\Omega,\hat{\lambda}_V)]r^{-2(\lambda_V+\lambda_\Omega+1)}\,h_VQ_V\\ &=\beta[(h_J/h_B)\,r^{\lambda_B-\hat{\lambda}_B}\,\chi(\lambda_B,\hat{\lambda}_J)\\ &\quad+(Q_J/Q_B)\,r^{\lambda_J-\hat{\lambda}_J}\,\chi(\lambda_J,\hat{\lambda}_B)]r^{-2(\lambda_B+\lambda_J+1)}\,h_BQ_B.\end{aligned} \tag{65}$$

Since $\lambda_V\neq\lambda_B$, and therefore $\lambda_V+\lambda_\Omega\neq\lambda_B+\lambda_J$, one may split Eq. (65) in two by requiring that the left and right sides vanish independently. This gives two equations formally identical with Eq. (59). The solution for each of them, i.e., $\hat{\lambda}_V(\lambda_V)$ and $\hat{\lambda}_B(\lambda_B)$, can then simply be read off from Fig. 2 or calculated from Eq. (61). Then Eq. (65) is an additional set of constraints. Since $\hat{\lambda}_V-\lambda_V\neq\hat{\lambda}_B-\lambda_B$ in this case, Eq. (51) is interpreted as a system of two equations for λ_V and λ_B. But the additional isotropic relation (56) renders this system overdetermined. It was checked that indeed no "eigenvalue" λ_B exists such that both relations between λ_V and λ_B could hold simultaneously.

Other ways of balancing terms and exponents are possible. For example, $\hat{\lambda}_V-\lambda_V=\hat{\lambda}_B-\lambda_B$ can be required in Eq. (64), leading to $\chi(\lambda_V,\hat{\lambda}_B)+(Q_B/Q_V)(h_V/h_B)\,\chi(\lambda_B,\hat{\lambda}_V)=0$, where Q_B/Q_V is a function of $\beta(\lambda_B)$. This gives one equation, say for $\beta_H=h_V/h_B$. Then the terms in Eq. (65) may be split into couples, and a balance is required for each couple individually. There are three ways of doing this, and in each case there are three equations: A balance of powers, which effectively involves only λ_V, λ_B, $\hat{\lambda}_V$, and $\hat{\lambda}_B$, and two equations involving couples of χ's as well as β/β_H. But there are only two unknowns to solve for: $\hat{\lambda}_V$ and $\hat{\lambda}_B$. There results a set of eigenproblems, each member being defined by a choice among the three ways of balancing and of argument triples such as $\{[S^V,B],[V,\Omega],[B,J]\}$, as in Sec. IV B. Again Eq. (56) is the overdetermining constraint, but Eq. (61) is no more valid. Unfortunately, there seem to be no solutions to these equations either.

V. FINAL REMARKS

The main results of this paper are the scaling exponent predictions (52) and (60), respectively, for the isotropic and ‘‘helical’’ part of the velocity structure function in neutral, incompressible, homogeneous turbulence. The corresponding vorticity scaling exponents are found as by-products. As expected, the anisotropic part of the velocity correlation grows faster with increasing distance, or in other words, the smaller scales are statistically more isotropic. The velocity structure function scaling exponent corresponding to isotropic correlations has been estimated experimentally at $\zeta_2 \approx 0.70$. The overshoot by the prediction, Eq. (52), proposed here is probably due to the neglect of cross correlation, rather than to the neglect of cumulants. Taking them into account may be expected to further increase the ‘‘anomaly’’ $\zeta_2 - 2/3$ due to deviations from Gaussianity. Since the present derivation deals directly with the generation of vorticity correlations by turbulent velocity, and since the positive anomaly found here has the sign of the experimentally observed anomaly, the result can be interpreted as an indication that, to no one’s surprise, a buildup of vorticity correlations is the mechanism behind the deviation from Kolmogorov scaling.

The observation of a range of positive β for which no isotropic MHD scaling could be found, may correspond to an observed qualitative *dichotomy* between low-β_T and high-β_T turbulences observed MHD flow. This effect is summarized, e.g., in Ref. [25] for astrophysical data as well as for DNS of MHD turbulence. It can crudely be described as follows. Statistically steady states fall into two distinct groups: either $\beta < 1$ but remaining $\mathcal{O}(1)$, or $\beta \gg 1$; transient states with intermediate β evolve to whichever steady state is ‘‘closer.’’ The expectation that there are two attracting values for the turbulent plasma β motivated a search for criteria that could select a couple of (well separated, positive) β eigenvalues from the spectrum given by Eq. (56), or by its ‘‘mirror image’’ given by Eq. (25) in [15]. It was found that, as far as isotropic scaling is concerned, the presented model allows for any ‘‘beta’’ measured in a turbulent MHD to be related to a unique value from either of the two β branches. A coupled solution for the isotropic and helical parts of the MHD correlation equations was sought, but no solution to the helical part could be found. This seems to indicate, again to no surprise, that cross correlations are important to the helical part of autocorrelations.

It seems worthwhile to attempt an extension of the presented approach to the problem of deriving the scaling exponents of anisotropic components of the neutral-turbulence velocity structure function. Expanding in spherical harmonics, one may use the results from [14] together with kinematic relations between the corresponding anisotropic components of vorticity and velocity autocorrelation to obtain analogs of Eq. (46). This should give an eigenvalue problem in each ‘‘anisotropy sector.’’ The resulting exponents should predict a decay of correlations faster than that of isotropic components.

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